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# BAER ORDERINGS WITH NONINVARIANT VALUATION RING

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ABSTRACT

We construct division algebras with involution containing a Baer ordering with noninvariant order ring. This gives a negative answer to a question of Holland, whether the order ring is always invariant under inner automorphisms. Furthermore, we give examples of any index. Previously, the only known examples of division algebras containing Baer orderings were of index $2^n$ or of index $p$ for $p$ a prime of the form $4m + 3$.

## 1. Introduction

Let $D$ be a finite-dimensional division algebra with involution. In [$H_1$] Holland defines a Baer ordering of $D$ and shows that associated to any Baer ordering is an order subring of $D$ which is a total valuation ring (see below for definitions). In [$H_1$] Holland raises the question whether the order ring is invariant under inner automorphisms of $D$. An affirmative answer would imply that associated to the ordering is a valuation on $D$. The existence of such a valuation could help to classify Baer orderings in much the same way as for orderings of commutative fields.

In this paper we give examples of a Baer ordering of a division algebra with involution having a noninvariant order ring. Thus we give a negative answer to Holland's question. An added feature of these examples is that we construct Baer orderings (with noninvariant order ring) of division algebras of every

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possible index, i.e., index $2^n$ for involutions of the first kind, and any index for involutions of the second kind. Previously the only known examples of division algebras containing Baer orderings were of index $2^n$ (with an involution of the first kind) [$H_3$, Th. 1.1] or index $p$ for $p$ a prime of the form $4m+3$ [$H_1$, Ex. 3.4].

The main tool in constructing these examples is valuation theory of division algebras. In Section 2 the relevant background information is given together with some results that will help to build our examples. The examples are then given in Section 3.

## 2. Preliminaries

Let $D$ be a division algebra with center $Z(D)=F$ and $*$ an involution on $D$. That is, $*$ is an antiautomorphism of $D$ with $*\circ *=1$. The involution $*$ is said to be of the first kind if $a^*=a$ for all $a\in F$; otherwise $*$ is said to be of the second kind. An element $d\in D$ is called symmetric (resp. skew) if $d^*=d$ (resp. $d^*=-d$). We will denote by $S(D)$ (resp. $Sk(D)$) the set of all symmetric (resp. skew) elements of $D$. The set $F_0=F\cap S(D)$ is called the symmetric subfield of $F$. So $F_0=F$ if $*$ is of the first kind, while $[F:F_0]=2$ if $*$ is of the second kind.

A *Baer ordering* of $D$ is a subset $B$ of $D$ such that

(i) $1\in B$ and $0\notin B$,

(ii) $B+B\subseteq B$,

(iii) $a^*Ba\subseteq B$ for all nonzero $a\in D$,

(iv) $B\cap -B=\varnothing$,

(v) $B\cup -B\supseteq S(D)-\{0\}$.

The idea behind this definition is that, while there is no ordering in the usual sense on a noncommutative division algebra finite dimensional over its center, if we restrict the order relation to the symmetric elements of a division algebra with involution then it is possible to have an ordering, for instance on the Hamilton quaternions **H**.

If $B$ is a Baer ordering on $D$ we define a partial order relation $>$ on $D$ by $a>b$ iff $a-b\in B$. Note that Holland's definition in [$H_1$] assumes that $B\cup -B=S(D)-\{0\}$. Given $B$ satisfying the above definition, $B\cap S(D)$ will be a Baer ordering according to Holland's definition. It will be convenient to use this slightly more general definition. We point out that there are other definitions of ordering on a division algebra with involution (cf. [C], [Cr], [$H_2$], [I]).

In order to establish Propositions 2.2 and 2.3 below, which allow us to build examples from more basic ones, we have found it convenient to introduce a parallel notion of Baer ordering which deals with skew elements. A subset $C$ of $D$ is called a *skew Baer ordering* of $D$ if

(i) $0 \notin C$,
(ii) $C + C \subseteq C$,
(iii) $a^*Ca \subseteq C$ for all nonzero $a \in D$,
(iv) $C \cap -C = \emptyset$,
(v) $C \cup -C \supseteq Sk(D) - \{0\}$.

Note that if $*$ is an involution of the second kind then $D$ has a nonzero central skew element $a$; if $B$ is a Baer ordering of $D$ then $aB$ is a skew Baer ordering of $D$. Conversely, if $C$ is a skew Baer ordering of $D$ then $\pm aC$ is a Baer ordering of $D$.

In the theory of ordered fields valuation theory has played an important role (cf. [P, §7], [L]). Similarly for the various types of orderings on division algebras with involution, connections have been found between orderings and valuations.

Let $D$ be a division algebra and $D^\bullet$ the set of nonzero elements of $D$. We work exclusively with a division algebra $D$ finite-dimensional over its center $Z(D)$. The *index* of $D$ is $\sqrt{[D : Z(D)]}$. A *valuation* on $D$ is a function $v : D^\bullet \to \Gamma$, where $\Gamma$ is a totally ordered abelian group, such that for all $a, b \in D^\bullet$,

(i) $v(ab) = v(a) + v(b)$,
(ii) $v(a + b) \geqq \min\{v(a), v(b)\}$ if $a + b \neq 0$.

For convenience we adjoin $\infty$ to $\Gamma$ with $\infty > \gamma$ for all $\gamma \in \Gamma$ and set $v(0) = \infty$. Let $\Gamma_D = v(D^\bullet)$, the value group of $v$; $V_D = \{a \in D \mid v(a) \geqq 0\}$, the valuation ring of $v$; $M_D = \{a \in D \mid v(a) > 0\}$, the maximal ideal of $V_D$; $U_D = V_D - M_D$, the group of units of $V_D$; and $\overline{D} = V_D/M_D$, the residue division ring of $v$. Let $\pi : V_D \to \overline{D}$ be the canonical homomorphism and write $\overline{a}$ for the image $\pi(a)$ of an element $a \in V_D$. If $E$ is a subdivision algebra of $D$ then $v|_E$ is a valuation on $E$ with valuation ring $V_E = V_D \cap E$. There are canonical inclusions $\Gamma_E \subseteq \Gamma_D$ and $\overline{E} \subseteq \overline{D}$.

Let $D$ be a division algebra with involution $*$ and suppose $v$ is a valuation on $D$. If $v(a^*) = v(a)$ for all $a \in D$ (i.e., $v \circ * = v$) then $v$ is said to be a $*$-valuation. If $*$ is of the first kind then $v$ is a $*$-valuation since by [W, Cor.] or [E, Cor. 1] the valuation $v|_F = (v \circ *)|_F$ has a unique extension to $D$. If $v$ is a $*$-valuation then the valuation ring $V_D$ of $v$ is closed under $*$, as is $M_D$, so there is an induced involution on $\overline{D}$, given by $\overline{u}^* = \overline{u^*}$.

Given a Baer ordering $B$ of $D$ the *order ring* of $B$ is defined to be

$$V_B = \{d \in D \mid r - dd^* \in B \text{ for some } r \in \mathbf{Q}\}.$$

This is a total subring of $D$ by [$H_1$, Th. 4.3]; that is, if $a \in D^\bullet$ then $a \in V_B$ or $a^{-1} \in V_B$. The question of whether $V_B$ is invariant under inner automorphisms of $D$ was raised in [$H_1$]. Only when this invariance holds will $V_B$ be the valuation ring of a valuation on $D$.

The orderings we will construct will be liftings of orderings on residue division algebras. The following lemma will allow the order ring of the lifted ordering to be determined.

LEMMA 2.1. *Let $D$ be a division algebra with involution $*$, $B$ a Baer ordering of $D$ and $v$ a $*$-valuation on $D$. Suppose that $\overline{B} = \pi(B \cap U_D)$ is a Baer ordering on $\overline{D}$. If $T$ is the order ring of $\overline{B}$ then $\pi^{-1}(T) \subseteq V_D$ is the order ring of $B$. If $T$ is not invariant under inner automorphisms then neither is $\pi^{-1}(T)$.*

PROOF. (Note that $v$ is trivial on $\mathbf{Q}$ since $\operatorname{char}(\overline{D}) = 0$.) Suppose $a \in V_B$. Then $r - aa^* \in B$ for some $r \in \mathbf{Q}$. If $v(a) < 0$ then $v(a^{-1}) = v((a^{-1})^*) > 0$, so

$$a^{-1}(r - aa^*)(a^{-1})^* = ra^{-1}(a^{-1})^* - 1 \in B.$$

Thus $\overline{ra^{-1}(a^{-1})^* - 1} = -\overline{1} \in \overline{B}$, a contradiction. Hence $v(a) \geqq 0$ and $\overline{r} - \overline{a}\overline{a}^* \in \overline{B}$ (or else $\overline{r} - \overline{a}\overline{a}^* = 0$; then $\overline{r+1} - \overline{a}\overline{a}^* \in \overline{B}$). Therefore $\overline{a} \in T$, and so $a \in \pi^{-1}(T)$. Conversely, if $a \in \pi^{-1}(T)$ then $\overline{r} - \overline{a}\overline{a}^* \in \overline{B}$ for some $r \in \mathbf{Q}$. Since $r - aa^*$ is symmetric, it has a sign, and so $\overline{r - aa^*} \in \overline{B}$ implies $r - aa^* \in B$. Hence $a \in V_B$. Therefore $V_B = \pi^{-1}(T)$. If $T$ is noninvariant then there is a $u \in U_D$ with $\overline{u}T\overline{u}^{-1} \neq T$. Then $V_B = \pi^{-1}(T) \neq u\pi^{-1}(T)u^{-1} = uV_Bu^{-1}$. ■

If $\mathscr{B}$ is a Baer ordering of $\overline{D}$, we call a Baer ordering $B$ of $D$ a *lifting* of $\mathscr{B}$ if $\overline{B} = \pi(B \cap U_D) = \mathscr{B}$. The above lemma applies whenever $B$ is a lifting of $\overline{B}$.

The next proposition will allow us to get all our desired examples (except of index $2^n$ with an involution of the second kind) from two basic constructions; one of index 2 and one of index $m$ for any odd integer $m$. Recall that a (finite-dimensional) valued division algebra $T$ with center $Z(T)$ is called *totally ramified* if $[T : Z(T)] = |\Gamma_T : \Gamma_{Z(T)}|$. It then follows from [S, pp. 21–22] that $\overline{T} = \overline{Z(T)}$.

PROPOSITION 2.2. *Let $A$ be a division algebra with involution $*_A$ (of either kind) with a Baer ordering $B$ and a skew Baer ordering $C$. Let $F_0$ be the symmetric subfield of $F = Z(A)$. Let $T$ be a division algebra with $Z(T) = K \supseteq F_0$, $[T : K] < \infty$, with involution $*_T$ of the first kind. Suppose $T$ has a valuation $v$ trivial on $F_0$ such that $T$ is totally ramified over $K$ and $\overline{T} = F_0$.*

*Let $D = A \otimes_{F_0} T$, with involution $* = *_A \otimes *_T$ (of the same kind as $*_A$). Then $D$ is a division algebra, and $D$ has a Baer ordering $B_D$ and a skew Baer ordering $C_D$. If the order ring of $B$ is noninvariant, then the order ring of $B_D$ is also noninvariant.*

PROOF. Before working with $D$, we look at $T$. Because $*_T$ is of the first kind, $v \circ *_T = v$. Let $R = \pi^{-1}(\overline{T}^{\bullet 2}) = (1 + M_T)U_T^2$, a normal subgroup of $T^\bullet$. Note that $1 + M_T$ is also a normal subgroup of $T^\bullet$. For $a, b \in T^\bullet$, $a \equiv b$ (mod $1 + M_T$) iff $v(a) = v(b) < v(a - b)$.

Since $T$ has an involution of the first kind it has order 2 in the Brauer group $\mathrm{Br}(K)$; consequently, by [JW$_2$, Cor. 6.10] or [PY, (3.19)] $\Gamma_T/\Gamma_K$ is an elementary abelian 2-group. Furthermore, from the theory of totally ramified division algebras (cf. [TW, §3]), there is a well-defined canonical pairing $\mathscr{C} : (\Gamma_T/\Gamma_K) \times (\Gamma_T/\Gamma_K) \to \{\pm 1\}$ given by $\mathscr{C}(v(a) + \Gamma_K, v(b) + \Gamma_K) = \overline{aba^{-1}b^{-1}}$. For notational convenience we view $\mathscr{C}$ as a map $\Gamma_T \times \Gamma_T \to \{\pm 1\}$. In particular, for $a, b \in T^\bullet$ with $v(b) \in \Gamma_K$ (e.g., if $v(b) \in 2\Gamma_T$), then $ab \equiv ba$ (mod $1 + M_T$).

Take any $x \in T^\bullet$. Since $v(x^{*_T}) = v(x)$ we have $x^{*_T} = ux$ for some $u \in U_T$. Then $x = (x^{*_T})^{*_T} = x^{*_T}u^{*_T} = uxu^{*_T} \equiv uu^{*_T}x$ (mod $1 + M_T$); so in $\overline{T} = \overline{K}$ (on which the residue involution is trivial) $\overline{u}\,\overline{u}^{*_T} = \overline{u}^2 = 1$; hence $\overline{u} = \pm 1$. That is, $x^{*_T} \equiv \pm x$ (mod $1 + M_T$). Define a function $\varepsilon : \Gamma_T \to \{\pm 1\}$ by

$$x^{*_T} \equiv \varepsilon(\gamma)x \pmod{1 + M_T} \quad \text{for any } x \in T^\bullet \text{ with } v(x) = \gamma.$$

It is easy to check that $\varepsilon(\gamma)$ depends only on $\gamma$ and not on the choice of $x$. The map $\varepsilon$ is not a group homomorphism. In fact,

$$(1) \qquad \varepsilon(\gamma + \delta) = \varepsilon(\gamma)\varepsilon(\delta)\mathscr{C}(\gamma, \delta) \qquad \text{for all } \gamma, \delta \in \Gamma_T.$$

From this it follows that

$$(2) \qquad \varepsilon(2\delta) = 1 \quad \text{and} \quad \varepsilon(\gamma + 2\delta) = \varepsilon(\gamma).$$

Now, we choose a special set of representatives $x_\gamma \in T^\bullet$ for each $\gamma \in \Gamma_T$ such that $v(x_\gamma) = \gamma$, and $x_0 = 1$. Define a function $\rho : \Gamma_T \times \Gamma_T \to U_T$ by $x_\gamma x_\delta = \rho(\gamma, \delta)x_{\gamma+\delta}$. The choice of $x_\gamma$ is to be made so that the following conditions hold for all $\gamma, \delta \in \Gamma_T$:

$$(3) \qquad \rho(\gamma, \delta) \equiv \pm 1 \pmod{R},$$

$$(4) \qquad \rho(\gamma, 2\delta) \equiv 1 \pmod{R}.$$

Here is a way of selecting the $x_\gamma$ to satisfy these conditions. First choose $\{\beta_i\}_{i \in I} \subseteq \Gamma_T$ mapping bijectively to a basis of the free $\mathbf{Z}/4\mathbf{Z}$-module $\Gamma_T/4\Gamma_T$.

For each $i \in I$ choose any $y_i \in T^\bullet$ with $v(y_i) = \beta_i$. Pick any total ordering of the index set $I$. Now, any $\gamma \in \Gamma_T$ is expressible uniquely in the form $\gamma = \Sigma_{i \in I} a_i \beta_i + 4\gamma'$, where $a_i \in \{0, 1, 2, 3\}$ and all but finitely many $a_i = 0$. Set $x_\gamma = \Pi_{i \in I} y_i^{a_i} \cdot z_\gamma^2$ where the $y_i^{a_i}$ terms in the product are written in ascending order of the $i$'s and $z_\gamma$ is chosen in $K$ so that $v(z_\gamma) = 2\gamma'$ (and $z_0 = 1$). This is possible since $2\Gamma_T \subseteq \Gamma_K$. If likewise $\delta = \Sigma_{i \in I} b_i \beta_i + 4\delta'$ so that $x_\delta = \Pi_{i \in I} y_i^{b_i} \cdot z_\delta^2$, then from the canonical pairing commutation rules we get

$$\rho(\gamma, \delta) \equiv \pm \prod_{i \in I_0} y_i^4 (z_\gamma z_\delta z_{\gamma+\delta}^{-1})^2 \pmod{1 + M_T}$$

where $I_0 = \{i \in I \mid a_i + b_i \geqq 4\}$. (If $\gamma + \delta = \Sigma_{i \in I} c_i \beta_i + 4(\gamma + \delta)'$ then $c_i = a_i + b_i$ if $a_i + b_i < 4$ and $c_i = a_i + b_i - 4$ if $a_i + b_i \geqq 4$.) The sign in the formula is determined by the canonical pairing and the exponents $a_i, b_i$. In particular, if $\delta \in 2\Gamma_T$ then all the $b_i$ are even, and the sign must be $+$ as $\mathscr{C}(2\alpha, \beta) = 1$ for all $\alpha, \beta \in \Gamma_T$. By writing $y_i^2 = u_i t_i$ with $u_i \in U_T$ and $t_i \in K$, we see that

$$\prod_{i \in I_0} y_i^4 (z_\gamma z_\delta z_{\gamma+\delta}^{-1})^2 = \left(\prod_{i \in I_0} u_i^2\right)\left(\prod_{i \in I_0} t_i z_\gamma z_\delta z_{\gamma+\delta}^{-1}\right)^2 \in R.$$

This proves (3) and (4).

We note one more property of $\rho$:

(5) $$\rho(\delta + \alpha, \delta + \alpha) \equiv \rho(\delta, \delta)\rho(\alpha, \alpha)\mathscr{C}(\alpha, \delta) \pmod{R}.$$

For this we compute $x_\delta x_\alpha x_\delta x_\alpha$ in two ways. First, mod $1 + M_T$

$$\begin{aligned} x_\delta x_\alpha x_\delta x_\alpha &\equiv \mathscr{C}(\alpha, \delta) x_\delta^2 x_\alpha^2 \\ &\equiv \mathscr{C}(\alpha, \delta)\rho(\delta, \delta) x_{2\delta} \rho(\alpha, \alpha) x_{2\alpha} \\ &\equiv \mathscr{C}(\alpha, \delta)\rho(\delta, \delta)\rho(\alpha, \alpha)\rho(2\delta, 2\alpha) x_{2(\delta+\alpha)}, \end{aligned}$$

but also

$$\begin{aligned} x_\delta x_\alpha x_\delta x_\alpha &= (\rho(\delta, \alpha) x_{\delta+\alpha})^2 \\ &\equiv \rho(\delta, \alpha)^2 \rho(\delta + \alpha, \delta + \alpha) x_{2(\delta+\alpha)} \pmod{1 + M_T}. \end{aligned}$$

Since $\rho(2\delta, 2\alpha), \rho(\delta, \alpha)^2 \in R$, from (4) and (3), this yields (5). Set $\rho'(\gamma, \delta) = 1$ if $\rho(\gamma, \delta) \in R$ and $\rho'(\gamma, \delta) = -1$ if $\rho(\gamma, \delta) \equiv -1 \pmod{R}$.

Finally, we need a function $\varphi : \Gamma_T \to \{\pm 1\}$ satisfying $\varphi(0) = 1$ and

(6) $$\varphi(\gamma + 2\alpha) = \varepsilon(\alpha)\mathscr{C}(\gamma, \alpha)\rho'(\alpha, \alpha)\varphi(\gamma) \qquad \text{for all } \alpha, \gamma \in \Gamma_T.$$

This can be obtained as follows: pick a set $\{\gamma_j\}_{j \in J}$ mapping bijectively to $\Gamma_T/2\Gamma_T$ with 0 the representative of $0 + 2\Gamma_T$. Define $\varphi(\gamma_j)$ arbitrarily in $\{\pm 1\}$, except $\varphi(0)=1$. Then any $\gamma\in\Gamma_T$ is expressible uniquely as $\gamma_j+2\sigma_\gamma$ for some $j\in J$ and $\sigma_\gamma\in\Gamma_T$. Define

$$\varphi(\gamma)=\varepsilon(\sigma_\gamma)\mathscr{C}(\gamma_j,\sigma_\gamma)\rho'(\sigma_\gamma,\sigma_\gamma)\varphi(\gamma_j).$$

Then for any $\alpha\in\Gamma_T$, using (1) and (5),

$$\begin{aligned}
\varphi(\gamma+2\alpha)&=\varphi(\gamma_j+2(\sigma_\gamma+\alpha))\\
&=\varepsilon(\sigma_\gamma+\alpha)\mathscr{C}(\gamma_j,\sigma_\gamma+\alpha)\rho'(\sigma_\gamma+\alpha,\sigma_\gamma+\alpha)\varphi(\gamma_j)\\
&=\varepsilon(\sigma_\gamma)\varepsilon(\alpha)\mathscr{C}(\sigma_\gamma,\alpha)\mathscr{C}(\gamma_j,\sigma_\gamma)\mathscr{C}(\gamma_j,\alpha)\rho'(\sigma_\gamma,\sigma_\gamma)\rho'(\alpha,\alpha)\mathscr{C}(\sigma_\gamma,\alpha)\varphi(\gamma_j)\\
&=\varepsilon(\alpha)\mathscr{C}(\sigma_\gamma,\alpha)^2\mathscr{C}(\gamma_j,\alpha)\rho'(\alpha,\alpha)\varphi(\gamma)\\
&=\varepsilon(\alpha)\mathscr{C}(\gamma_j,\alpha)\rho'(\alpha,\alpha)\varphi(\gamma)\\
&=\varepsilon(\alpha)\mathscr{C}(\gamma,\alpha)\rho'(\alpha,\alpha)\varphi(\gamma)
\end{aligned}$$

as $\gamma\equiv\gamma_j \bmod 2\Gamma_T$, proving (6).

We now turn to $D$. Since $A\otimes_{F_0}\overline{T}\cong A$ is a division ring, [M, Th. 1] shows that $D=A\otimes_{F_0}T$ is a division ring and the valuation $v$ on $T$ (together with the trivial valuation on $F_0$ and $A$) extends to a valuation on $D$ with $\Gamma_D=\Gamma_T$ and $\overline{D}=A$. Moreover, the residue involution on $\overline{D}$ is $*_A$. Identify $T$ with its image $1\otimes T$ in $D$. Let $P_1=B$ and $P_{-1}=C$. Define $B_D$ and $C_D$ by

$$B_D=\bigcup_{\gamma\in\Gamma_D}\varphi(\gamma)\pi^{-1}(P_{\varepsilon(\gamma)})x_\gamma \quad\text{and}\quad C_D=\bigcup_{\gamma\in\Gamma_D}\varphi(\gamma)\pi^{-1}(P_{-\varepsilon(\gamma)})x_\gamma.$$

We check that $B_D$ is a Baer ordering of $D$. The verification that $C_D$ is a skew Baer ordering is analogous. Note that the union for $B_D$ is disjoint, and that $(1+M_D)B_D\subseteq B_D$ since this is true for each term in the union. This shows that $a+b\in B_D$ whenever $a,b\in B_D$ and $v(a)\neq v(b)$. But if $v(a)=v(b)$, we get $a+b\in B_D$ since $P_{\varepsilon(v(a))}+P_{\varepsilon(v(a))}\subseteq P_{\varepsilon(v(a))}$. Thus $B_D+B_D\subseteq B_D$. Also $1\in B_D$ and $0\notin B_D$ as $\varphi(0)=1$, $x_0=1$, *and* $\bar{1}\in B=P_{\varepsilon(0)}$ as $\varepsilon(0)=1$. We have $B_D\cap -B_D=\emptyset$ as $P_{\varepsilon(\gamma)}\cap -P_{\varepsilon(\gamma)}=\emptyset$. For $a\in S(D)-\{0\}$ write $a=ux_\gamma$ where $\gamma=v(a)$ and $u\in U_D$. Then as conjugation by $x_\gamma$ induces the identity on $\overline{D}$,

$$ux_\gamma=a=a^*=x_\gamma^*u^*\equiv\varepsilon(\gamma)x_\gamma u^*\equiv\varepsilon(\gamma)u^*x_\gamma \pmod{1+M_D}.$$

Hence $\bar{u}^*=\varepsilon(\gamma)\bar{u}$ in $\overline{D}=A$. If $\varepsilon(\gamma)=1$, $\bar{u}^*=\bar{u}$, so $\pm\bar{u}\in B=P_1$, so $\pm a\in\varphi(\gamma)\pi^{-1}(P_{\varepsilon(\gamma)})x_\gamma\subseteq B_D$. Likewise if $\varepsilon(\gamma)=-1$, then $\bar{u}^*=-\bar{u}$, so $\pm\bar{u}\in C=P_{-1}$, so again $\pm a\in B_D$. Thus $S(D)-\{0\}\subseteq B_D\cup -B_D$.

Finally we check $a^*B_Da \subseteq B_D$. Take $d = ux_\gamma \in B_D$ with $u \in U_D$; so $\varphi(\gamma)\bar{u} \in P_{\varepsilon(\gamma)}$. Write $a = tx_\alpha$ with $t \in U_D$. Then, mod $1 + M_D$

$$\begin{aligned} a^*da &= x_\alpha^* t^* u x_\gamma t x_\alpha \\ &\equiv \varepsilon(\alpha) x_\alpha t^* u t x_\gamma x_\alpha \\ &\equiv t^* u t \varepsilon(\alpha) \mathscr{C}(\gamma, \alpha) x_\gamma x_\alpha^2 \\ &\equiv t^* u t \varepsilon(\alpha) \mathscr{C}(\gamma, \alpha) \rho(\alpha, \alpha) \rho(\gamma, 2\alpha) x_{\gamma+2\alpha}. \end{aligned}$$

We have, by (3) and (4), $\rho(\alpha, \alpha) = \rho'(\alpha, \alpha) r_1$ and $\rho(\gamma, 2\alpha) = r_2$ with $r_1, r_2 \in R$. So $\bar{r}_1, \bar{r}_2 \in \bar{T}^{\bullet 2} = F_0^{\bullet 2}$. Write $a^*da = sx_{\gamma+2\alpha}$. To see $a^*da \in B_D$ we must check $\varphi(\gamma + 2\alpha)\bar{s} \in P_{\varepsilon(\gamma+2\alpha)}$. We have, using (6),

$$\begin{aligned} \varphi(\gamma + 2\alpha)\bar{s} &= (\varepsilon(\alpha)\mathscr{C}(\gamma, \alpha)\rho'(\alpha, \alpha)\varphi(\gamma))(\bar{t}^*\bar{u}\bar{t}\varepsilon(\alpha)\mathscr{C}(\gamma, \alpha)\rho'(\alpha, \alpha)\bar{r}_1\bar{r}_2) \\ &= \bar{r}_1 \bar{r}_2 \bar{t}^* \varphi(\gamma)\bar{u}\bar{t}. \end{aligned}$$

This lies in $P_{\varepsilon(\gamma+2\alpha)}$ as $\varphi(\gamma)\bar{u} \in P_{\varepsilon(\gamma)} = P_{\varepsilon(\gamma+2\alpha)}$ by (2), using the multiplicative closure property of $B$ and $C$ (which implies in particular, $F_0^{\bullet 2} P_{\varepsilon(\gamma+2\alpha)} \subseteq P_{\varepsilon(\gamma+2\alpha)}$). Lemma 2.1 yields the assertion about the order rings, completing the proof. ■

Remark. One choice of $T$ for Proposition 2.2 is

$$\left(\frac{x_1, x_2}{K}\right) \otimes_K \cdots \otimes_K \left(\frac{x_{2n-1}, x_{2n}}{K}\right)$$

where

$$K = F_0(x_1, \ldots, x_{2n})$$

and the $x_i$ algebraically independent over $F_0$ (cf. [JW$_1$, Ex. 2.7]). But other choices of $T$ exist, including some $T$ which are not isomorphic to a tensor product of quaternion algebras (cf. [CW, §4]).

The next proposition will allow us to take a division algebra (of index $2^n$) with involution of the first kind which contains a Baer ordering with noninvariant order ring and construct a division algebra with involution of the second kind (also of index $2^n$) which also contains a Baer ordering with noninvariant order ring.

Proposition 2.3. *Let $A$ be an $F$-central division algebra with involution $*_A$ of the first kind, $B$ a Baer ordering of $A$ and $C$ a skew Baer ordering of $A$. Let $K = F(z)$, where $z$ is transcendental over $F$, and $\sigma$ the $F$-automorphism of $K$ given by $\sigma(z) = -z$. Let $* = *_A \otimes \sigma$, an involution of the second kind on*

*$D = A \otimes_F K$. Then there is a Baer ordering $B_D$ of $D$, whose order ring is noninvariant if the order ring of $B$ is noninvariant. Also, $D$ has a skew Baer ordering, and the index of $D$ is equal to the index of $A$.*

PROOF. Let $v$ be the $z$-adic valuation of $K$. Using the trivial valuation on $A$ we see $v$ extends to $D$ with $\overline{D} = A$ by [M, Th. 1]. The residue involution on $\overline{D}$ is $*_A$. Note that every nonzero element $d$ of $D$ is expressible uniquely in the form $d = uz^n$ where $v(u) = 0$ and $n \in \mathbf{Z}$. Also, $d^* = d$ iff $u^* = u$ for $n$ even or $u^* = -u$ for $n$ odd.

Let $B_D \subseteq D$ be defined by

$$B_D = \bigcup_{n \in \mathbf{Z}} (\pi^{-1}(B)z^{4n} \cup \pi^{-1}(-B)z^{4n+2} \cup \pi^{-1}(C)z^{4n+1} \cup \pi^{-1}(-C)z^{4n+3}).$$

This is a disjoint union. Clearly $1 \in B_D$, $0 \notin B_D$ and $B_D \cap -B_D = \emptyset$ since this holds for $B$ and $C$. Also $B_D \cup -B_D \supseteq S(D) - \{0\}$ as $B \cup -B \supseteq S(A) - \{0\}$ and $C \cup -C \supseteq Sk(A) - \{0\}$. As each of $\pi^{-1}(\pm B)$, $\pi^{-1}(\pm C)$ is closed under addition and $(1 + M_D)B_D \subseteq B_D$, we have $B_D + B_D \subseteq B_D$ as in the proof of Proposition 2.2. for $d = uz^m \in D^\bullet$ with $v(u) = 0$ and $\varepsilon = wz^n \in D^\bullet$, we have

$$e^*de = (wz^n)^*(uz^m)(wz^n) = (-1)^n(w^*uw)z^{m+2n}.$$

Since $\overline{u} \in B$ implies $\overline{w}^*\overline{u}\,\overline{w} \in B$ and $\overline{u} \in C$ implies $\overline{w}^*\overline{u}\,\overline{w} \in C$, it is easy to check in all cases that if $d \in B_D$ then $e^*de \in B_D$. Hence $B_D$ is a Baer ordering of $D$ with $\overline{B_D} = \pi(B_D \cap U_D) = B$. The information on the order rings then follows from Lemma 2.1. Let

$$C_D = \bigcup_{n \in \mathbf{Z}} (\pi^{-1}(C)z^{4n} \cup \pi^{-1}(-C)z^{4n+2} \cup \pi^{-1}(B)z^{4n+1} \cup \pi^{-1}(-B)z^{4n+3}).$$

A calculation similar to the one just given for $B_D$ shows $C_D$ is a skew Baer ordering of $D$. Finally it is clear that $D$ and $A$ have the same index. ■

## 3. The examples

In this section we construct Baer orderings with noninvariant order ring in division algebras of any index. We first construct an example of index 2 with an involution of the first kind, then an example of index $m$ for $m$ any odd integer. We then use these with Propositions 2.2 and 2.3 to produce all the desired examples. The examples given below will be built from symbol algebras. If $F$ is a field, $\omega$ a primitive $n$-th root of unity in $F$ and $a, b \in F^\bullet$, the symbol algebra $A_\omega(a, b; F)$ is the $n^2$-dimensional central simple $F$-

algebra generated by elements $i, j$ subject to the relations $i^n = a$, $j^n = b$ and $ji = \omega ij$. For $\omega = -1$ $(n = 2)$, $A_{-1}(a, b; F)$ is the ordinary quaternion algebra $((a,b)/F)$.

We now give the index 2 example. Let $L$ be a field with an archimedean ordering, $x$ and $y$ commuting indeterminates over $L$, let

$$F = L(x, y) \quad \text{and} \quad D = \left(\frac{1 + x, -y}{F}\right).$$

If $i$ and $j$ are the usual generators of $D$ then $i^2 = 1 + x$, $j^2 = -y$ and $k = ij = -ji$, so $k^2 = (1 + x)y$. The involution $*$ we use on $D$ is the "$j$-involution", that is, the standard involution followed by conjugation by $j$. Thus $i^* = i$, $j^* = -j$, $k^* = k$ and $*$ is of the first kind. Let $v$ be the $y$-adic valuation on $F$, so that $\bar{F} = L(x)$. By [JW$_2$, Ex. 4.3] $D$ is a division algebra and $v$ extends to $D$ so that $\Gamma_D = \frac{1}{2}\mathbf{Z}$ and $\bar{D} = L(x)(\sqrt{1 + x})$. Furthermore, $v(i) = 0$, $v(j) = v(k) = \frac{1}{2}$, and $v(y) = 1$. The valuation $v$ is a $*$-valuation since $*$ is of the first kind. The induced involution on $\bar{D}$ is the identity since $i^* = i$. Let $\sigma$ be the nonidentity $\bar{F}$-automorphism of $\bar{D}$. Conjugation by $j$ (or $k$) induces $\sigma$ on $\bar{D}$. The $x$-adic valuation ring of $L(x)$ has two extensions to $\bar{D} = L(x)(\sqrt{1 + x})$ since $1 + x$ is a 1-unit with respect to this valuation ring; call these valuation rings $T_1$ and $T_2$. Note that $\sigma(T_1) = T_2$ and $\sigma(T_2) = T_1$. Let $V_i = \pi^{-1}(T_i)$. Each $V_i$ is a total valuation ring of $D$. However, $V_i$ is noninvariant since $jV_1j^{-1} = V_2$ and $jV_2j^{-1} = V_1$. We will construct a Baer ordering of $D$ whose order ring is $V_1$.

Since $T_1$ is a valuation ring of the field $\bar{D}$ with residue field $L$ having an archimedean ordering, there is an ordering $P$ of $\bar{D}$ with order ring $T_1$. Then $\sigma(P)$ is another ordering of $\bar{D}$ (with order ring $T_2$). Also, let $Q$ be an ordering of $\bar{F} = L(x)$ such that $1 + x \in -Q$ (for example, let $Q$ be an ordering whose order ring is $L[x]_{(x+2)}$). Note that for any $u \in \bar{D}^{\bullet}$, $u\sigma(u) \in Q$ since $u\sigma(u)$ has the form $f^2 - (1 + x)g^2$ with $f, g \in L(x)$. Define $B$ by

$$B = \bigcup_{n \in \mathbf{Z}} (\pi^{-1}(P)y^{2n} \cup \pi^{-1}(\sigma(P))y^{2n+1} \cup \pi^{-1}(Q)y^{2n}k \cup \pi^{-1}(-Q)y^{2n+1}k).$$

Observe that this is a disjoint union since $\pi^{-1}(P)$, $\pi^{-1}(\sigma(P))$, $\pi^{-1}(Q) \subseteq U_D$, so each term in the union has a different value. For a skew Baer ordering of $D$ first let $R$ be any ordering of $F$ with $1 + x \in -R$ and $y \in R$, then set $C = Rj$.

**Theorem 3.1.** *With the notation above, $B$ is a Baer ordering of the quaternion algebra $D$ with respect to the involution $*$ of the first kind. The* *order ring of B is $V_1$ which is noninvariant. Furthermore, C is a skew Baer ordering of D.*

PROOF. It is clear that $1 \in B$ and $0 \notin B$. Furthermore, $B \cap -B = \varnothing$ since $P \cap -P = \varnothing$, $\sigma(P) \cap -\sigma(P) = \varnothing$, and $Q \cap -Q = \varnothing$. We verify that $B$ satisfies the remaining conditions to be a Baer ordering.

$B + B \subseteq B$. Take $a, b \in B$. If $v(a) < v(b)$ then $a + b \in B$ since $(1 + M_D)B \subseteq B$, as each of $\pi^{-1}(P)$, $\pi^{-1}(\sigma(P))$, $\pi^{-1}(Q)$ is closed under multiplication by $1 + M_D$. If $v(a) = v(b)$ then $a + b \in B$ since $\pi^{-1}(P)$, $\pi^{-1}(\sigma(P))$, $\pi^{-1}(Q)$ are closed under addition.

$a^*Ba \subseteq B$. Let $\alpha = ty^l$ with $v(t) = 0$ and $l \in \mathbf{Z}$. Now, every $a \in D^\bullet$ has the form either (i) $a = uy^s$ (if $v(a) \in \mathbf{Z}$) or (ii) $a = juy^s$ (if $v(a) \in \frac{1}{2} + \mathbf{Z}$) with $v(u) = 0$ and $s \in \mathbf{Z}$. So we have either

$$\text{(i)}\quad a^*\alpha a = u^*tuy^{2s+l},$$

or

$$\text{(ii)}\quad a^*\alpha a = u^*(jtj^{-1})uy^{2s+l+1}.$$

If $\alpha \in \pi^{-1}(P)y^{2n}$ (i.e., $\bar{t} \in P$ and $l = 2n$) then as $\bar{D}$ is commutative, $*$ is trivial on $\bar{D}$, $P\bar{D}^{\bullet 2} \subseteq P$, $\sigma(P)\bar{D}^{\bullet 2} \subseteq \sigma(P)$ and conjugation by $j$ induces $\sigma$ on $\bar{D}$, we have $\overline{u^*tu} = \bar{t}\bar{u}^2 \in P$ and $\overline{u^*(jtj^{-1})u} = \sigma(\bar{t})\bar{u}^2 \in \sigma(P)$. So in case (i) $a^*\alpha a \in \pi^{-1}(P)y^{2(s+n)} \subseteq B$ and in case (ii) $a^*\alpha a \in \pi^{-1}(\sigma(P))y^{2(s+n)+1} \subseteq B$. Similarly, if $\alpha \in \pi^{-1}(\sigma(P))y^{2n+1}$ (so $\bar{t} \in \sigma(P)$ and $l = 2n + 1$) we find (as $\sigma^2 = 1$) either (i) $a^*\alpha a \in \pi^{-1}(\sigma(P))y^{2(s+n)+1} \subseteq B$, or (ii) $a^*\alpha a \in \pi^{-1}(P)y^{2(s+n+1)} \subseteq B$.

Now consider $\alpha$ of the form $\alpha = ty^lk$ with $v(t) = 0$ and $l \in \mathbf{Z}$. For $a$ of type (i) or (ii) as above we have either

$$\begin{aligned}\text{(i)}\quad a^*\alpha a &= y^su^*ty^lkuy^s \\ &= [u^*t(kuk^{-1})]y^{2s+l}k,\end{aligned}$$

or

$$\begin{aligned}\text{(ii)}\quad a^*\alpha a &= y^su^*j^*ty^lkjuy^s \\ &= u^*(-j)t(-j)(kuk^{-1})y^{2s+l}k \\ &= [-u^*(jtj^{-1})(kuk^{-1})]y^{2s+l+1}k.\end{aligned}$$

Now suppose $\alpha \in \pi^{-1}(Q)y^{2n}k$ (i.e., $\bar{t} \in Q \subseteq \bar{F}$ and $l = 2n$). Then, as $\bar{D}$ is commutative, conjugation by $k$ induces $\sigma$ on $\bar{D}$, norms from $\bar{D}^\bullet$ to $\bar{F}^\bullet$ lie in $Q$, and $Q \cdot Q \subseteq Q$, in case (i) $\overline{u^*t(kuk^{-1})} = \bar{t}\bar{u}\sigma(\bar{u}) \in Q$, so $a^*\alpha a \in \pi^{-1}(Q)y^{2(s+n)}k \subseteq B$; in case (ii) since conjugation by $j$ induces the identity on

$\bar{F}$, $\overline{-u^*(jtj^{-1})(kuk^{-1})} = -\bar{t}\bar{u}\sigma(\bar{u}) \in -Q$, so $a^*\alpha a \in \pi^{-1}(-Q)y^{2(s+n)+1}k \subseteq B$. Similarly, if $\alpha \in \pi^{-1}(-Q)y^{2n+1}k$ (so $\bar{t} \in -Q$ and $l = 2n+1$), then either (i) $a^*\alpha a \in \pi^{-1}(-Q)y^{2(s+n)+1}k \subseteq B$, or (ii) $a^*\alpha a \in \pi^{-1}(Q)y^{2(s+n+1)}k \subseteq B$. Thus in all cases, $a^*Ba \subseteq B$.

$B \cup -B \supseteq S(D) - \{0\}$. Take $s = s^* \in D^\bullet$. We have $s = f + gi + hk$ with $f, g, h \in F$. Since $v(f+gi) \in \mathbf{Z}$ and $v(hk) \in \frac{1}{2} + \mathbf{Z}$, these values are not equal. If $v(s) = v(f+gi) < v(hk)$, then $s = uy^l$ with $v(u) = 0$. Since $\bar{u} \in \bar{D}^\bullet = P \cup -P = \sigma(P) \cup -\sigma(P)$, we have

$$\pm s \in \bigcup_{n \in \mathbf{Z}} (\pi^{-1}(P)y^{2n} \cup \pi^{-1}(\sigma(P))y^{2n+1}) \subseteq B.$$

If $v(s) = v(hk) < v(f+gi)$, then $s \equiv hk \pmod{1 + M_D}$, so $s = ty^lk$, with $v(t) = 0$ and $\bar{t} \in \bar{F}$. Since $\bar{F}^\bullet = Q \cup -Q$ we again have $\pm s \in B$. This completes the proof that $B$ is a Baer ordering of $D$.

By the construction of $B$ it is clear that $\bar{B} = P$ is a Baer ordering of $\bar{D}$ (as $*$ is trivial on $\bar{D}$). Since $P$ has order ring $T_1$, $B$ has order ring $\pi^{-1}(T_1) = V_1$ by Lemma 2.1.

Finally, we show $C$ is a skew Baer ordering of $D$. Note that $0 \notin C$ since $0 \notin R$; $C + C \subseteq C$ since $R + R \subseteq R$; and $C \cap -C = \varnothing$ since $R \cap -R = \varnothing$. As $R \cup -R = F^\bullet$, $C \cup -C = F^\bullet j = Sk(D) - \{0\}$. Last, to show $a^*Ca \subseteq C$ for $a \in D^\bullet$, write $a = f + gi + hj + lk$ with $f, g, h, l \in F$. Let $N = a^*jaj^{-1}$. Since $*$ is the standard involution followed by conjugation by $j$, $N$ is the quaternionic norm of $a$, i.e.,

$$N = f^2 - g^2(1+x) + h^2y - l^2(1+x)y.$$

Because $-(1+x)$, $y \in R$, we have $N \in R$. Take any $\delta j \in C$, so $\delta \in R$. Then $a^*(\delta j)a = \delta(a^*jaj^{-1})j = \delta Nj \in Rj = C$. Thus, $a^*Ca \subseteq C$. Therefore, $C$ is a skew Baer ordering of $D$. ■

REMARK. This $D$ and $V_1$ were originally constructed by the second author as an example of a division algebra with a $c$-ordering with noninvariant order ring (see [C, correction]).

We next consider division algebras of odd index. Our construction was inspired by the index $p$ example given by Holland in [$H_1$, Ex. 3.4].

THEOREM 3.2. *For any odd integer $m$ there is a division algebra $S$ of index $m$ with involution (of the second kind) with a Baer ordering with noninvariant order ring. Furthermore, $S$ also has a skew Baer ordering.*

PROOF. Let $\omega$ be a primitive $m$-th root of unity in $\mathbf{C}$, $x$ and $y$ commuting indeterminates and set

$$F_0 = \mathbf{Q}(\omega + \overline{\omega})(x, y), \quad F = F_0(\omega), \quad S = A_\omega(1 + x, y; F).$$

($\overline{\omega}$ is the complex conjugate of $\omega$.) Then $S$ is a central simple $F$-algebra of dimension $m^2$. Let $K = F(\sqrt[m]{y})$. Then $K/F_0$ is Galois of degree $2m$. Let $\mathscr{G}(K/F_0)$ be the Galois group of $K/F_0$. If $\varphi \in \mathscr{G}(K/F_0)$ is defined by $\varphi(\sqrt[m]{y}) = \omega \sqrt[m]{y}$ and $\varphi|_F = 1$, and $\sigma$ is complex conjugation extended to $K$ by $\sigma(\sqrt[m]{y}) = \sqrt[m]{y}$ and $\sigma(x) = x$, then $\sigma\varphi\sigma = \varphi^{-1}$. Furthermore $F$ is the fixed field of $\varphi$ and $S$ is the cyclic algebra $(K, \varphi, 1 + x)$. We define an involution on $S$ as follows. Write $S = \bigoplus_{i=0}^{m-1} Kt^i$ with $t^m = 1 + x$ and $tct^{-1} = \varphi(c)$ for $c \in K$. We then define $*$ on $S$ extending $\sigma$ by

$$\left(\sum_{i=0}^{m-1} c_i t^i\right)^* = \sum_{i=0}^{m-1} \varphi^i(\sigma(c_i)) t^i.$$

A straightforward calculation shows $*$ is an involution (of the second kind) on $S$ which fixes $F_0$.

Let $v$ be the $y$-adic valuation on $F_0$. Then $v$ extends uniquely to $K$ such that $K/F$ is totally ramified with $\Gamma_K = (1/m)\mathbf{Z}$, $\Gamma_{F_0} = \mathbf{Z}$, $F/F_0$ unramified, $\overline{F} = \mathbf{Q}(\omega)(x)$ and $\overline{F_0} = \mathbf{Q}(\omega + \overline{\omega})(x)$. Since $K/F$ is totally ramified and $1 + x$ is a unit of $V_F$ with $[\overline{F}(\sqrt[m]{1+x}) : \overline{F}] = m$, [JW$_2$, Ex. 4.3] shows $S$ is a division algebra and $v$ extends to $S$ with $\Gamma_S = (1/m)\mathbf{Z}$ and $\overline{S} = \overline{F}(\sqrt[m]{1+x})$. Since $(v \circ *)|_F = v|_F \circ \sigma|_F = v|_F$ (as $v|_{F_0}$ extends uniquely to $F$), $v \circ * = v$ by [W, Cor.] or [E, Cor. 1]. Hence $v$ is a $*$-valuation. Furthermore, $v$ is smooth, in the terminology of [H$_1$]. To see this, let $\gamma \in \Gamma_S$. As $|\Gamma_S : \Gamma_{F_0}| = m$ is odd, $\Gamma_S = 2\Gamma_S + \Gamma_{F_0}$. Thus $\gamma = 2v(c) + v(a) = v(cc^*a)$ for some $c \in S$, $a \in F_0$. Thus if $s = cc^*a$ then $s$ is symmetric and $v(s) = \gamma$. Also, if $A_t$ denotes the automorphism of $\overline{S}$ induced by conjugation by any $t \in S^\bullet$, we have $A_s \circ * = A_{cc^*} \circ * = A_c \circ * \circ A_c^{-1}$. Hence $s \in v^{-1}(\gamma)$ is smooth symmetric, and so by [H$_1$, Cor. 3.3], given a Baer ordering $\mathscr{B}$ of $\overline{S}$, there is a Baer ordering $B$ of $S$ with $\overline{B} = \mathscr{B}$.

To determine $V_B$ we determine the order ring of $Q$ and apply Lemma 2.1. The order ring of $P$ is the $x$-adic valuation ring $T'$ of $\mathbf{Q}(\omega + \overline{\omega})(x)$. The valuation ring $T'$ extends uniquely to the $x$-adic valuation ring $T''$ of $\mathbf{Q}(\omega)(x) = \overline{F}$. Thus $T''$ is the order ring of $Q \cap \overline{F}$. As $1 + x$ is a 1-unit in $T''$, $T''$ extends in $m$ ways to $\overline{S} = \overline{F}(\sqrt[m]{1+x})$. One of these, say $T$, is the order ring of $Q$. Note that if $\tau : \overline{S} \to \overline{S}$ is defined by $\tau(\sqrt[m]{1+x}) = \omega \sqrt[m]{1+x}$ and $\tau|_{\overline{F}} = 1$ then the extensions of $T''$ to $\overline{S}$ are $T, \tau(T), \ldots, \tau^{m-1}(T)$. If $i$ and $j$ are the usual generators of $S$ with $i^m = 1 + x$ and $j^m = y$ then conjugation by $j$ induces $\tau$ on $\bar{S}$. By Lemma 2.1, $V_B = \pi^{-1}(T)$. Thus $jV_Bj^{-1} = \pi^{-1}(\tau(T)) \neq \pi^{-1}(T) = V_B$. Therefore $V_B$ is noninvariant.

As the involution on $S$ is of the second kind, $S$ also has a skew Baer ordering, by the comment after the definition of skew Baer ordering in Section 2. ■

The two preceding examples and Propositions 2.2 and 2.3 are all that is needed to produce division algebras with involution of any index which contain Baer orderings with nonivariant order ring.

THEOREM 3.3. *For any positive integer n there are division algebras of index $2^n$ with Baer ordering with noninvariant order ring. Examples exist with involution of the first kind, and also with involution of the second kind.*

PROOF. Let $D$ be the quaternion division algebra of Theorem 3.1, and let $F = Z(D)$. Let $T$ be a totally ramified valued division algebra of index $2^{n-1}$ with center $K \supseteq F$, with involution $*_T$ of the first kind, satisfying $\bar{T} = F$. For example, we could take $k = F(x_1, \ldots, x_{2n-2})$ with the $x_i$ algebraically independent over $F$, and set

$$T = \left(\frac{x_1, x_2}{K}\right) \otimes_K \cdots \otimes_K \left(\frac{x_{2n-3}, x_{2n-2}}{K}\right),$$

with $*_T$ the tensor product of usual involutions on each quaternion factor. The standard valuation on the iterated Laurent series field $F((x_1))\cdots((x_{2n-2}))$ restricts to a valuation on $K$ which has a totally ramified extension to $T$ (cf. [$JW_1$, Ex. 2.7]). (If $n = 1$, set $T = K = F$.) Let $E = T \otimes_F D$ and $E' = E \otimes_K K(z)$, where $z$ is transcendental over $K$. So each of $E$ and $E'$ has index $2^n$. Set $*_E = *_T \otimes *_D$ and $*_{E'} = *_E \otimes \sigma$, where $\sigma$ is the $K$-automorphism of $K(z)$ given by $\sigma(z) = -z$. Then $*_E$ is an involution on $E$ of the first kind and $*_{E'}$ is an involution on $E'$ of the second kind. Theorem 3.1 and Propositions 2.2 and 2.3 show that $E$ and $E'$ each have a Baer ordering with respect to the given involution with noninvariant order ring; they each have a skew Baer ordering, as well. ■

THEOREM 3.4. *For any integer $l > 1$ with $l$ not a 2-power, there is a division algebra A of index l with involution of the second kind containing a Baer ordering with noninvariant order ring.*

PROOF. Write $l = 2^r m$ where $m > 1$ is odd. Let $S$ be the division algebra of index $m$ given by Theorem 3.2. Let $T$ be a valued division algebra with involution of the first kind, with $T$ totally ramified of index $2^r$ over its center $K \supseteq F_0$, with $\bar{T} = F_0$. For example, take $T$ as in the proof of Theorem 3.3. If $r = 0$, set $T = F_0$. Set $A = T \otimes_{F_0} S$ with involution $*_A = *_T \otimes *_S$; Theorem 3.2 and Proposition 2.2 show $A$ has the desired properties. ■

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